Math 261B Thurs. Oct. 15

Representations of G $\qquad G \curvearrowright V \overset{\cong}{\text{is}} K^n$

or coaction $\rho : V \to V \otimes \mathcal{O}(G)$

$T \subset B \subset G$ $\qquad$ torus $\subset$ Borel $\subset G$

$X = X(T)$ = weight lattice $\qquad R_+$ = pos. roots

$$V = \bigoplus_{\lambda \in X} V_\lambda \quad \text{(as } T \text{ module)} \qquad V_\lambda = \text{weight space}$$

$\alpha \in R \rightsquigarrow$ root $SL_2 \to G$ $\qquad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$\cup$

$U_\alpha \cong \mathbb{G}_a$, with coordinate x

$U_\alpha \curvearrowright V$ by $V \xrightarrow[\rho]{} V \otimes \mathcal{O}(G) \longrightarrow V \otimes \mathcal{O}(U_\alpha) \overset{\shortparallel}{K[x]}$

$\mathcal{O}(G) \longrightarrow \mathcal{O}(U_\alpha)$

$G \hookleftarrow U_\alpha$

$v_0 \in V_\lambda$ $\qquad v_0 \underset{\rho}{\mapsto} \sum_{i \geq 0} v_i \otimes x^i$

v_0 on RHS really is v_0, since e is $x = 0$

$a \cdot v_0 = \sum a^i v_i$

v_i has weight $\lambda + i\alpha$

Aside (char $K=0$) $\quad u_k = k!\, v_k$

$x, y \in U_\alpha \qquad x v_0 = x u_0 = \sum_k \frac{x^k}{k!} u_k$

$y \cdot x = y + x \qquad y x v_0 = \sum_k \frac{(x+y)^k}{k!} u_k = \sum_{i,j} \frac{x^i}{i!} \frac{y^j}{j!} u_{i+j}$

$\qquad\qquad = y \sum_i \frac{x^i}{i!} u_i$

equate coefficients of $x^i/i!$

$$y \cdot u_i = \sum_j \frac{y^j}{j!} u_{i+j}$$

In basis $u_m, u_{m-1}, \dots u_0$ (m largest s.t. $u_m \neq 0$)

x acts as $\boxed{\exp x \cdot \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & 0 & 1 & \\ \vdots & \vdots & 0 & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}}$

Over $\mathbb{C}$ this is just

$$\begin{tikzcd} \mathrm{Lie}(U_\alpha) \arrow[r, "\exp"] \arrow[d] & U_\alpha \curvearrowright V \arrow[d] \\ \mathrm{Lie}(G) \arrow[r] & G \curvearrowright V \end{tikzcd}$$

$\mathrm{Lie}(U_\alpha) = \mathbb{C}$, $\quad U_\alpha \cong (\mathbb{C}, +)$

$$x \cdot v_i = \sum \binom{i+j}{i} x^j v_{i+j}$$

If char $K = p$, could have $v_i = 0$ $\quad v_{i+j} \neq 0$ if $\binom{i+j}{i} \equiv 0 \pmod p$

Notice if $\lambda + i\alpha$ isn't a weight of V for any $i>0$, then

$U_\alpha v_\lambda = v_\lambda$

$\rightarrow$ There's always some $\eta : X \to \mathbb{Z}$ s.t. $\langle \eta, \alpha \rangle > 0$ for $\alpha \in R_+$

E.g. $\eta = \sum_{R_+} \alpha^\vee$ works

$\Rightarrow$ $\exists$ a weight λ of V s.t. no $\lambda + i\alpha$ is a weight for any $\alpha \in R_+$, $i>0$ $\Rightarrow$ $v \in V_\lambda$ is U invariant U = unipotent rad. of B

V has a U-invariant weight vector $B = T \ltimes U$

W acts on weights of V $\Rightarrow$ $\dim V_\lambda$ is W-invariant

λ weight $\Rightarrow$ $s_\alpha \lambda$ for all $\alpha \in R_+$

$= \lambda - \langle \alpha^\vee, \lambda \rangle \alpha$

$\Rightarrow$ λ has to have $\langle \alpha^\vee, \lambda \rangle \geq 0$ for all $\alpha \in R_+$

or equivalently $\langle \alpha_i^\vee, \lambda \rangle \geq 0$ $\alpha_i \in \Delta$.

λ is <u>dominant</u>

If space of U invariant weight vectors in V is 1-dim'l, then that space generates an irreducible submodule of V.

Ex. $SL_2 \curvearrowright K^2$, $\curvearrowright \mathcal{O}(K^2) = K[x,y] \supset K[x,y]_d$

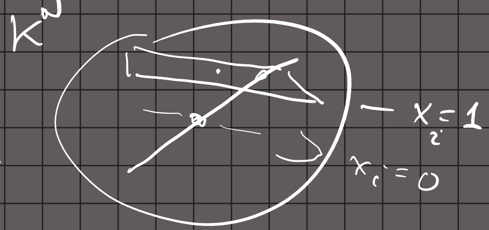
$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is $y \mapsto ax + y$ U invariant vectors $= f(x)$

x^d (weight $d \geq 0$)

Geometric construction : flag variety G/B is a projective algebraic variety.

Ex. $G = GL_n$: B = upper Δ'ars

$G/B = \{ \text{flags } 0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset K^n, \ \dim F_d = d \}$



Standard flag :

$0 \subset E_1 \subset E_2 \subset \cdots$

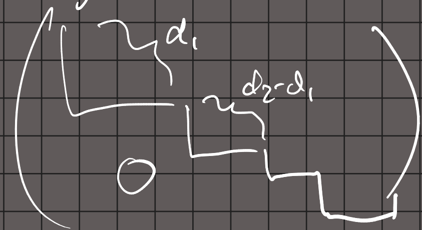
$E_d = \langle e_1, \dots, e_d \rangle$

$g \in GL_n$ fixes $E_\cdot \iff g \in B$

$$\begin{pmatrix} * & & * \\ & * & \\ 0 & & * \end{pmatrix}$$

GL_n acts transitively on flags.

Subgroups $P \supseteq B$ are block upper triangular matrices



P = stabilizer of $0 \subset E_{d_1} \subset E_{d_2} \subset \cdots$

G/P = partial flags F. with dimensions d_i

V $\quad v \in V_\lambda$ $\quad U$-invariant

$Kv \cong K_\lambda$ as T-module

$B \to B/U \cong T \curvearrowright K_\lambda$

$\lambda \in X \mapsto K_\lambda$ is a B-module via $B \to T$

$G \times_B K_\lambda = \mathcal{L}_\lambda$

$B \curvearrowright G \times K_\lambda$, $\quad K_\lambda = K = \mathbb{A}^1$

$b(g, v) = (gb^{-1}, bv)$

$(g, v) \mapsto g \cdot v$ is B-invariant

$(G \times K_\lambda)/B \longrightarrow G/B$

$(g, v) \longmapsto gB$

fibers are copies of K

is a rank 1 vector bundle, or line bundle over G/B.

$$\begin{array}{ccccc} (g, w) & \mapsto & p \in G \times_B K_\lambda & \mapsto & gw \in V \\ w = \mathbb{C}\cdot v & & & & gw = \mathbb{C}\, gv \end{array}$$

$$K_\lambda \otimes K_\lambda^* \xrightarrow{\sim} K, \qquad K_\lambda^* = K_{-\lambda}, \qquad K = K_0$$

$$\begin{tikzcd} \mathcal{L}_\lambda \arrow[r, "\pi"] \arrow[d] & V \arrow[r, dashed, "f"] & K \\ G/B & & \end{tikzcd}$$

$f \in V^*$ ⇢ $\pi^* f$ on $\mathcal{L}_\lambda$ linear on fibers, so it's a global section of $\mathcal{L}_{-\lambda}$

Dual bundle of $\mathcal{L}_\lambda$ is $\mathcal{L}_{-\lambda}$

← has a U_--invariant vector of weight $-\lambda$

$$V^* \longrightarrow H^0(G/B, \mathcal{L}_{-\lambda})$$

non-zero, hence injective if V^* is irreducible.

$E \in G/B$ fixed by B $\quad U_- E$ is open in G/B ← (irreducible)

$E = 1B/B$

$\Downarrow$ dense

G/B

$H^0(G/B, \mathcal{L}_{-\lambda})$ has a 1-dimensional space of U_- invariant elements (of weight $-\lambda$)

(for any dominant λ)

(also has 1 U_+ invariant element of weight $w_0(-\lambda)$

changes R_- to R_+))